A penalty scheme and policy iteration for nonlocal HJB variational inequalities with monotone drivers

Yufei Zhang
Joint work with Christoph Reisinger

Mathematical Institute, University of Oxford

Oxford, 4 September 2018

Outline

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, we consider the value function:

$$u(t,x) = \sup_{\tau} \sup_{\alpha} \underbrace{\mathbb{E}^{t,x}\Big[\int_t^{\tau} e^{-r(s-t)} \ell(\alpha_s, X_s^{\alpha,t,x})\, ds + e^{-r(\tau-t)} \xi(\tau, X_\tau^{\alpha,t,x})\Big]}_{:=\mathcal{L}_{t,\tau}^{\alpha,t,x}[\xi(\tau, X_\tau^{\alpha,t,x})]},$$

over all admissible control processes α and stopping times $\tau \in [t, T]$, subject to the the controlled SDE:

$$\begin{aligned} dX_s^{\alpha,t,x} =& b(\alpha_s, X_s^{\alpha,t,x})\, ds + \sigma(\alpha_s, X_s^{\alpha,t,x})\, dW_s \\ &+ \eta(\alpha_s, X_s^{\alpha,t,x}, e)\, \tilde{N}(ds, de),\ s \in [t, \tau]; \quad X_t^{\alpha,t,x} = x, \end{aligned} \tag{1}$$

and the terminal payoff:

$$\xi(\tau, X_\tau^{\alpha,t,x}) = \zeta(\tau, X_\tau^{\alpha,t,x}) 1_{t \le \tau < T} + g(X_T^{\alpha,t,x}) 1_{\tau = T}.$$

A generalized mixed optimal stopping and control problem:

$$u(t,x) = \sup_{\tau} \sup_{\alpha} \mathcal{E}^{\alpha,t,x}_{t,\tau}[\xi(\tau, X^{\alpha,t,x}_{\tau})].$$

For example,

- American options in an imperfect market;
- optimal investment with (nonlinear) stochastic utilities;
- robust pricing under model uncertainty, e.g.

$$\mathcal{E}^{\alpha,t,x}_{t,\tau}[\cdot] = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{t,x}_{\mathbb{Q}}[\cdot] \quad \text{or} \quad \mathcal{E}^{\alpha,t,x}_{t,\tau}[\cdot] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{t,x}_{\mathbb{Q}}[\cdot].$$

In this talk, we assume $\mathcal{E}^{\alpha,t,x}_{t,\tau}[\cdot]$ is induced by a backward SDE.

Consider the value function

$$u(t,x) = \sup_{\tau}\sup_{\alpha} \mathcal{E}^{\alpha,t,x}_{t,\tau}[\xi(\tau, X^{\alpha,t,x}_{\tau})] := \sup_{\tau}\sup_{\alpha} Y^{\alpha,t,x}_{t,\tau},$$

where the process $(Y^{\alpha,t,x}_{s,\tau})_{t\le s\le \tau}$ satisfies the following backward SDE: $Y^{\alpha,t,x}_{\tau,\tau} = \xi(\tau, X^{\alpha,t,x}_{\tau})$, and $s \in [t,\tau]$,

$$\begin{aligned} -dY^{\alpha,t,x}_{s,\tau} =& f(\alpha_s, X^{\alpha,t,x}_s, Y^{\alpha,t,x}_{s,\tau}, Z^{\alpha,t,x}_{s,\tau}, K^{\alpha,t,x}_{s,\tau})ds - Z^{\alpha,t,x}_{s,\tau} dW_s \\ & - K^{\alpha,t,x}_{s,\tau}\, \tilde{N}(ds,de), \end{aligned}$$

and $X^{\alpha,t,x}$ is given by the controlled jump-diffusion process (1).

Remark

We consider a continuous driver f, which is monotone in y, i.e.,

$$(y-y')(f(\alpha,x,y,z,k) - f(\alpha,x,y',z,k)) \le \mu |y-y'|^2,$$

for some $\mu \in \mathbb{R}$, and Lipschitz continuous in z and k.
The classical linear expectation case corresponds to the additive driver $f(\alpha,x,y,z,k) \equiv \ell(\alpha,x) - ry$.

Consider the HJB variational inequality (HJBVI):

$$\min\left\{u(\mathbf{x})-\zeta(\mathbf{x}), u_t+\inf_{\alpha\in\mathbf{A}}\left(-L^\alpha u-f(\alpha,x,u,(\sigma^\alpha)^T Du, B^\alpha u)\right)\right\}=0$$

for $\mathbf{x}\in\mathcal{Q}_T=(0,T]\times\mathbb{R}^d$ and $u(0,x)=g(x)$ for $x\in\mathbb{R}^d$.
The operators $L^\alpha := A^\alpha + K^\alpha$ and B^α are given by:

$$A^\alpha\phi(\mathbf{x})=\frac{1}{2}\mathrm{tr}(\sigma^\alpha(x)(\sigma^\alpha(x))^T D^2\phi(\mathbf{x}))+b^\alpha(x)\cdot D\phi(\mathbf{x}),$$

$$K^\alpha\phi(\mathbf{x})=\int_E\left(\phi(t,x+\eta^\alpha(x,e))-\phi(\mathbf{x})-\eta^\alpha(x,e)\cdot D\phi(\mathbf{x})\right)\nu(de),$$

$$B^\alpha\phi(\mathbf{x})=\int_E\left(\phi(t,x+\eta^\alpha(x,e))-\phi(\mathbf{x})\right)\gamma(x,e)\,\nu(de),$$

where ν is the singular measure on $E=\mathbb{R}^n\setminus\{0\}$ and $\mathbf{A}$ is compact.

For any given parameter $\rho \geq 0$, consider the penalized problem:

$$u_t^\rho + \inf_{\alpha \in \mathbf{A}} \left(- L^\alpha u^\rho - f(\alpha, x, u^\rho, (\sigma^\alpha)^T Du^\rho, B^\alpha u^\rho) \right) - \rho(\zeta - u^\rho)^+ = 0,$$

for $(t, x) \in \mathcal{Q}_T$, and $u(0, x) = g(x)$ for $x \in \mathbb{R}^d$.

- It holds as $\rho \to \infty$ that

$$(\zeta - u^\rho)^+ \to 0,$$

  thus u^ρ converges to the solution u of the HJBVI as $\rho \to \infty$.
- Moreover, we have $u^{\rho_1} \leq u^{\rho_2} \leq u$, for any $0 \leq \rho_1 \leq \rho_2$.

Theorem (Convergence rate of the value function)

Suppose the obstacle ζ is Lipschitz continuous in x and Hölder continuous in t with exponent $\mu \in (0,1]$, then we have

$$0 \le u(\mathbf{x}) - u^\rho(\mathbf{x}) \le C_0 \rho^{-\min(\mu, \frac{1}{2})}, \quad \mathbf{x} \in \bar{\mathcal{Q}}_T.$$

If we further assume $\zeta \in C_b^{1,2}(\bar{Q}_T)$, then we have

$$0 \le u(\mathbf{x}) - u^\rho(\mathbf{x}) \le C_0/\rho, \quad \mathbf{x} \in \bar{\mathcal{Q}}_T.$$

To approximate the free boundary:

$$\Gamma = \{\mathbf{x} \in \bar{\mathcal{Q}}_T \mid u(\mathbf{x}) = \zeta(\mathbf{x})\},$$

we suppose the estimate $0 \leq u(\mathbf{x}) - u^\rho(\mathbf{x}) \leq C_0 \rho^{-\mu}$ holds for some constants $C_0 > 0$ and $\mu \in (0, 1]$, and define for each $\rho > 0$ the set

$$\Gamma_\rho = \{\mathbf{x} \in \bar{\mathcal{Q}}_T \mid \zeta(\mathbf{x}) - C_0 \rho^{-\mu} \leq u^\rho(\mathbf{x}) \leq \zeta(\mathbf{x})\}.$$

It holds that $\Gamma \subset \Gamma_\rho$ for all $\rho > 0$, and

$$\lim_{\rho \to \infty} d_{\mathcal{H}}(\Gamma_\rho \cap K, \Gamma \cap K) = 0,$$

for any given compact subset $K \subset \bar{\mathcal{Q}}_T$.

Discretize the penalized equation:

$$\inf_{\alpha\in\mathbf{A}} \left(u_t^\rho-(A^\alpha+K^\alpha)u^\rho-f(\alpha,x,u^\rho,(\sigma^\alpha)^T Du^\rho,B^\alpha u^\rho)\right)-\rho(\zeta-u^\rho)^+ = 0$$

by a semi-implicit monotone scheme: for $n=0,\dots,N-1$,

$$\inf_{\alpha\in\mathbf{A}} \Bigg(\frac{U_i^{n+1}-U_i^n}{\Delta t}-A_h^\alpha U_i^{n+1} - K_h^\alpha U_i^n - \bar{f}(\alpha,x_i,U_i^{n+1},\Delta U_i^n, B_h^\alpha U_i^n) - \rho(\zeta(t_{n+1},x_i)-U_i^{n+1})^+\Bigg) = 0, \quad i\in\mathbb{Z}^d,$$

with monotone approximations $A_h^\alpha \approx A^\alpha$, $K_h^\alpha \approx K^\alpha$, $B_h^\alpha \approx B^\alpha$, and a monotone numerical flux $\bar{f}$ for the nonlinearity of f on Du.

- Stability in sup-norm: CFL condition independent of the penalty parameter ρ.
- Convergence: for each fixed $\rho \geq 0$, the numerical solution converges to the solution of the penalized equation uniformly on compact sets as $h \to 0$.

Remark

Well-posedness: construct Lipschitz approximations of the monotone driver.

Given $u^n \in \mathbb{R}^M$, find $u \in \mathbb{R}^M$ by solving

$$
\begin{aligned}
0 =& \mathcal{G}_h^{n+1}[u]_i \\
=& \inf_{\alpha \in \mathbf{A}} \Bigg(\frac{u_i - u_i^n}{\Delta t} - A_h^\alpha u_i - K_h^\alpha u_i^n - \bar{f}(\alpha, x_i, u_i, \Delta u_i^n, B_h^\alpha u_i^n) \\
& - \rho(\zeta(t_{n+1}, x_i) - u_i)^+ \Bigg), \quad i = 1, \ldots, M.
\end{aligned}
$$

- Generalized policy iteration can be applied if the driver f admits a "weak" partial derivative $\partial_y^o f$ in y such that:

$$
f(\cdot,\cdot,y+h,\cdot,\cdot) - f(\cdot,\cdot,y,\cdot,\cdot) - \partial_y^o f(\cdot,\cdot,y+h,\cdot,\cdot)h = o(h),
$$

  which is satisfied by piecewise differentiable functions, convex/concave functions and more generally semismooth functions.

- Given the current iterate $u^{(k)}$, we compute

$$\alpha_i^{(k+1)} \in \arg\min_{\alpha \in \mathbf{A}} \mathcal{G}_h^{n+1}[u^{(k)}]_i, \quad \forall i = 1, \dots, M,$$

  and find the next iterate $u^{(k+1)}$ by solving a linear system.
- The iterates $\{u^{(k)}\}$ converge superlinearly to the solution u of $\mathcal{G}_h^{n+1}[u] = 0$ in a neighbourhood of u, i.e.,

$$\|u^{(k+1)} - u\| = \mathcal{O}(\|u^{(k)} - u\|).$$

- The sets of optimal controls

$$\mathbf{A}_{u^{(k)}} := \prod_{i=1}^{M} \arg\min_{\alpha \in \mathbf{A}} \mathcal{G}_h^{n+1}[u^{(k)}]_i$$

  converge in terms of the Hausdorff metric as $k \to \infty$.

Consider a risk-free asset and a risky asset

$$dS_t = S_{t^-}[b\,dt + \sigma\,dW_t + (1 \wedge |e|)\,\tilde{N}(dt, de)],$$

on $(\Omega, \mathcal{F}, \mathbb{P})$.
An investor with initial wealth x at t can control their wealth process $X^{\alpha,t,x}$ by choosing the percentage α_t of wealth held in the risky asset, and also the duration of the investment τ, which leads to the terminal payoff $\xi_\tau^{\alpha,t,x} = g(X_\tau^{\alpha,t,x})$.

For given parameters $r, R, \kappa_1, \kappa_2 > 0$, we consider the following ambiguity in the market:

- uncertainty in the discount rate: $\mathcal{B}_t$ is a class of adapted processes $\beta = (\beta_s)_{s\in[t,T]}$ valued in $[r, R]$;
- uncertainty in the Brownian motion and the random jump source: $\mathcal{M} = \{\mathbb{Q} \sim \mathbb{P} \mid \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t} = M_t^{\pi,\ell}\}$ such that

$$dM_t^{\pi,\ell} = M_{t-}^{\pi,\ell}\left(\pi_t dW_t + \int_E \ell_t(e)\,\tilde{N}(de, dt)\right); \quad M_0^{\pi,\ell} = 1,$$

where (π, ℓ) are predictable processes satisfying $|\pi_t| \le \kappa_1$ and $0 \le \ell_t(e) \le \kappa_2(1 \wedge |e|)$.

Maximize the performance in the worst-case scenario:

$$u_*(t,x) = \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t} \inf_{\beta \in \mathcal{B}_t, \mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \Big[\exp\big(- \int_t^\tau \beta_s \, ds \big) \xi_\tau^{\alpha,t,x} \Big],$$

which corresponds to a HJBVI with a concave driver:

$$\min \Big\{ \inf_{\alpha \in [0,1]} \big(u_t - L^\alpha u - r u^- + R u^+ + \alpha \kappa_1 \sigma |x u_x| + \kappa_2 B_*^\alpha u \big), \\ u(t,x) - g(x) \Big\} = 0, \quad x \in (0,T] \times \mathbb{R}.$$

We can also consider the best-case scenario:

$$u^*(t,x) = \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t} \sup_{\beta \in \mathcal{B}_t, \mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \Big[\exp\big(- \int_t^\tau \beta_s \, ds \big) \xi_\tau^{\alpha,t,x} \Big],$$

which corresponds to a HJBVI with a convex driver.

- $g(x) = 1 - 2e^{-2x}$.
- Lévy measure $\nu(de) = \frac{1}{|e|} \exp(-\mu|e|)de$ on $\mathbb{R}$.
- Model parameters :

b	σ	μ	r	R	κ_1	κ_2	T	x_0
0.1	0.2	6	0.02	0.04	0.2	0.5	1	1

Table: Parameters for the optimal investment problem under ambiguity.

- Choose $\Delta t = \mathcal{O}(h)$ for a consistent approximation. Set the threshold of policy iteration as 10^{-10}.
- Approximation error: $\mathcal{O}(h) + \mathcal{O}(\Delta t)$.

	h	1/40	1/80	1/160	1/320	1/640
$\rho = 10^3$	(a)	4	4	4	4	5
	(b)	0.7292780	0.7292918	0.7292987	0.7293021	0.7293038
	(c)			2.004	2.004	2.002
$\rho = 16 \cdot 10^3$	(a)	4	4	4	5	4
	(b)	0.7293262	0.7293271	0.7293275	0.7293277	0.7293278
	(c)			2.004	2.004	2.009

Table: Numerical solutions of the value function u_* for the worst-case senario. Shown are: (a) the maximal number of iterations among all time points; (b) the numerical solutions $U_{\rho,h}$ at (T, x_0); (c) the rate of increments $(U_{\rho,2h} - U_{\rho,4h})/(U_{\rho,h} - U_{\rho,2h})$.

- Perform computations with the mesh size $h = 1/640$.
- A first-order monotone convergence.

ρ		10^3	4×10^3	16×10^3	64×10^3
u^*	(a)	0.75071151	0.75071215	0.75071231	0.75071235
	(b)			3.9998	4.0006
u_*	(a)	0.72930381	0.72932303	0.72932783	0.72932903
	(b)			4.0016	3.9976

Table: Numerical results of the value functions u^* and u_* with different penalty parameters. Shown are: (a) the numerical solutions U_ρ at (T, x_0); (b) the rate of increments $(U_{\rho/4} - U_{\rho/16})/(U_\rho - U_{\rho/4})$.

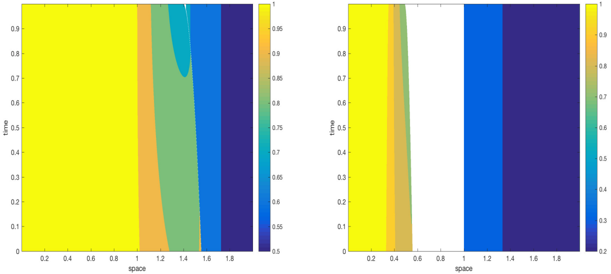

Figure: Feedback control strategies with $\rho = 16 \times 10^3$ for the best-case scenario (left) and the worst-case scenario (right), where the early stopping region is white.

- Construct the solution and free boundary of HJBVIs with monotone drivers from a sequence of penalized equations, for which the penalization error is estimated.
- Establish the well-posedness and convergence of semi-implicit monotone schemes for the penalized equation.
- Propose policy iteration with local superlinear convergence for solving the discrete equation.
- C. Reisinger, and Y. Zhang, *A penalty scheme and policy iteration for nonlocal HJB variational inequalities with monotone drivers*, preprint, arXiv:1805.06255 [math.NA], 2018.

Thank You!